MINOR STUDIES FROM THE PSYCHOLOGICAL LABORATORY OF CLARK UNIVERSITY.

Communicated by EDMUND C. SANFORD.

XVIII. COUNTING AND ADDING.

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The simple arithmetical processes offer an attractive field for psychological study, not only because they form a definite group of highly developed and characteristically human mental activities, but, also, because a clear understanding of them might be expected to throw light upon a branch of education to which many weary hours are devoted. The aim of the present study is to examine some aspects of two of these processes, as they exist in the minds of educated adults.

I.

COUNTING.

The most fundamental of all the arithmetical processes both mathematically and psychologically is counting. This has been described by Hall and Jastrow as "the matching, or pairing, or approximative synchronization of the terms of two series of events in consciousness."[1] These two series are the series of number names and the series of objects to be counted. In order that counting may begin it is necessary that the series of number names shall first be well established mentally—in fact shall have become actually automatic, so that when once started it may be trusted to run off of itself, leaving the attention free to supervise the pairing of the terms in this series with those in the other. It is therefore natural that a child in learning to count first masters the number names as a purely linguistic feat and only later comes to apply them in actual enumeration.[2] The series of number names once mastered, the child

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[1] Hall and Jastrow: Studies of Rhythm, *Mind*, XI, 1886, 55-62.

[2] This suffers an exception in the case of the numbers one, two and three, the groups corresponding to which are so early and so easily discriminated that the names for them may be learned like the names of things; and the ability to count objects within that range may thus come at practically the same time as the mastery of the number names. A little girl of five years to whom the writer gave a number of lessons in counting knew at that age, without any specific instruction, the number names one, two, three, four, five, eight and nine. She could count three objects correctly and sometimes four, but could do nothing beyond this.

is in a position to learn, and gradually to associate with each number name, the special qualities of the group to which it corresponds; so that, for example, "nine" shall upon occasion mean (or be associated with) three times three, or four plus five, or the square root of eighty-one, or any of the other relations in which it may occur.

In educated adults the number series has, of course, long passed the learning stage and become extremely automatic, and the readiest point for study of it is in its application under various conditions. Two such have been especially regarded in this study: (1) the counting of groups of visible objects at different distances from the eye, and (2) the counting of series of irregularly recurring clicks.[1]

In general it may be expected that anything that interferes with the equable succession of either the inner or the outer series, or that interferes with their co-ordination will be a hindrance to counting. In the first of the cases to be considered the trouble lay chiefly in the outer series, in the second in the habitual uniformity of the inner series.

The Counting of Visual Objects. The objects to be counted were small labels, colored black, one-half by three-fourths of an inch in size and one inch apart, and were placed in a horizontal row on a level with the eye of the observer. They were counted first singly, from left to right, then from right to left, then in groups of twos, threes, fours and fives, from left to right and from right to left as before. The distances at which the observer was stationed were, in different series, nine, fifteen, twenty-four and thirty-six feet. Two observers worked at the experiments, their task being not only to do the counting but also to note what might be discoverable with regard to the psycho-physiology of the process. The number of labels counted each time was about thirty, the observer never knowing the exact number beforehand.

It was observed in the counting, as might be expected, that the members of the number series were usually spoken, if not aloud, at least mentally, as the eye came to rest on one label (or group of labels) after another, the sudden stop and some slight muscular emphasis bringing the co-ordination to consciousness.

The counting of a series of small black labels arranged as in this case against a smooth light background, with no points of orientation except the spots themselves, brings into relief every method and device of the organism to keep the place as the eye passes along the line. One observer for example found himself taking notice of marks in the wainscoting three feet below the

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[1] This sort of counting was suggested by Binet and Henri as a test of attention. *L'Année psychologique*, II, 1895, 446.

labels as points of orientation. At a distance of nine feet the labels could be counted with ease and certainty; at fifteen feet there was some difficulty which was increased still further at the greater distances. At thirty-six feet there was a tendency to pass beyond the label desired, and this became a fruitful source of errors in the count. Another source of error was the involuntary wink.

An interesting practical question is that of the most economical manner of counting. Counting by twos was felt to be a little more difficult than counting by ones, but the time of counting a given total of labels was, of course, decidedly lessened. The angular motion of the eyes was twice as great, and thus more clearly conscious, but more care was required to keep the place accurately, and upon this depends the counter's feeling of confidence in the accuracy of his count. Counting by threes is for the same reason more difficult than by twos. In counting by fours the observer usually found himself taking two twos, and in counting by fives a two and a three. The following little table, giving the results for one of the observers, based upon total counts of from 1,200 to 2,500 for each distance, shows that the time for counting a given total decreases as the size of the group increases, but less rapidly; while the accuracy is greatest in the double and quadruple group. As accuracy is usually the chief desideratum in counting, the count by twos would seem preferable under circumstances similar to those of these experiments.

Counting by	Average time in seconds	Average of errors
ones	20.4	5.0
twos	14.0	2.0
threes	11.5	3.8
fours	10.7	2.3
fives	9.6	3.5

The results for the second observer agree fully with these. A greater degree of certainly was felt in counting from right to left than in the reverse direction, but the reason of this was not discovered.

The Counting of an Irregular Series of Clicks. The apparatus for providing these clicks consisted of two pendulums of slightly different rate arranged with mercury contacts at the central points of their arcs. These were so wired that the transit of either pendulum across its drop of mercury caused the armature of a small electro-magnet to strike against its cores with a single sharp click while the sound of the back stroke was, by suitable means, eliminated. The movements of the armature were inscribed at the same time upon a revolving drum and thus objectively recorded. When both the pendu-

lums were set in motion at the same time they produced a set of clicks separated by intervals of varying length, the whole set recurring, after a number of seconds, according to a fixed system. The rate of the pendulums was such that, if the clicks had occurred at equal intervals, the counting would have been extremely easy.

The task of the observer was to count the clicks to the best of his ability. This he did silently and without seeing the pendulums or the recording stylus. Practice was continued until he could count with such a degree of certainty that, if he made an error, he could tell just where it occurred and explain the cause. The difficulty lay in co-ordinating the inner series of number names, which has a simple and uniform advance, with the procession clicks which were now crowded close together in pairs and now single and more widely separated. About thirty clicks were counted in each group, the observers, of course, being ignorant of the actual number given. Usually from fifeeen to seventeen groups were counted at a sitting. Four gentlemen, students in the University, served as subjects. They usually made many errors at first, but after five or six days felt certain of their counts and were justified by the record. Co-ordinating two such series involves either the complete giving up the serial character of the inner series or the remodeling of its rhythm to fit such as may be found in the outer series. Learning to count successfully meant, on the subjects' part, learning to hold themselves securely in readiness to record *(i. e.,* to count) instantly the one click or two clicks as they occurred, and then to be ready for the next. After some experience the observer mastered the nature of the recurrences in the series of clicks and knew pretty well when to expect either one click or a pair.

The point of chief interest in this experiment is the emphasis which it places upon the simply rhythmic character of the automatic inner series. The counting automatism, like all others, tends naturally toward a uniform periodicity, and special training is necessary for the acquirement of any other. This training in the case of counting is training of the mechanisms of inner speech, and is no doubt largely motor.

Counting in General. The psychical counting mechanism is in its operation essentially like the common mechanical counters in use in laboratories and elsewhere. It is so arranged that with a proper succession of (generally) like excitations to action it will bring forth its characteristic series of numerical symbols—the series of number names in inner or outer speech. In the excitations that cause it to operate an important element, if not the chief one, is voluntary movement, which serves as a signal for its action, or possibly in some cases, causes the

latter by a direct overflow of energy into the speech centre. The importance of this factor is evident in the need of special fixation of the eyes in visual counting, and of accommodating the rhythms in counting clicks. It is evident also in the inclination to touch with the finger actual objects to be counted or tell them off on the fingers, or by pencil strokes, or in some other way, in ordinary counting. Indeed it may be questioned whether counting would be possible at all without at least some trace of voluntary movement to put the psychophysic mechanism in operation. It is certain that counting would be immensely more difficult and carry with it less confidence of accuracy.

II.

ADDING.

I have sought to get light upon the process of adding in two ways; first by getting a number of gentlemen to add columns of figures for me that I might study their methods of adding both objectively and by the aid of their introspections, and secondly, by accurately timing the addition of certain pairs of numbers in the case of myself and another subject.

On the Adding of Columns of Figures. In this first group of tests eight gentlemen assisted me, with one exception students in Clark University, though not of the mathematical department. Each usually added for an hour at a time with a few minutes pause at the end of the first half hour. The figures to be added were presented in columns of twenty-seven figures each, fifteen columns to a sheet. The adding was done aloud and the partial sums were set down by the experimenter (generally myself) as fast as they were announced by the subject. At the end of each column both subject and experimenter went over together the series of sums and the subject explained any peculiarities of the results. In this way nearly 200 columns were added.

The most striking feature of the experiment was the variety of procedure in adding. Some of the subjects, and these were probably among the most rapid and least liable to error, simply added digit to digit all the way up the column, rarely if ever going out of their way to form combinations of the figures lying just ahead before adding them to the sum already reached. Others, on the contrary, rarely missed an opportunity to assist themselves in this way. As an example of the methods, I may cite the work of subjects C and E, which is typical of the extreme types.

C added 810 digits and announced 519 results. During the adding he used 213 combinations, 171 of which were of two digits, and 42 were of three, with an occasional one of four

digits where they were small. The combinations used were as follows:

Digits grouped to form	No. of times.	Digits grouped to form	No. of times.
3	4	10	36
4	6	11	18
5	18	12	27
6	10	13	13
7	10	14	9
8	16	15	4
9	21		

As to the number of each, there were 8(5+7) combinations; 6(2+8); 6(3+2); 6(4+1); 5(8+4); 5(4+7), etc. This shows in some measure the relative frequency of grouping certain digits.

E added 840 digits and announced the same number of results. He named the first digit and added each digit separately and was so rapid in announcing his partial sums that it was difficult for the experimenter to record them.

The combined results for all the subjects were as follows: Of the 5,295 digits added, 2,975 were added singly and 974 combinations were used. Of the latter, 914 were of two figures and 60, of three or more figures. Of the frequency with which the groups were used, the following table is a representation:

20.9%	were	groups	the	sums	of	which equaled	10.
13.6%	"	"	"	"	"	"	9.
10.4%	"	"	"	"	"	"	5.
9.5%	"	"	"	"	"	"	8.
9.4%	"	"	"	"	"	"	11.
8.4%	"	"	"	"	"	"	7.
8.3%	"	"	"	"	"	"	6.
5.9%	"	"	"	"	"	"	12.
.etc.							etc.

As shown above, the number decreases as the interval increases, either way, from ten; five, possibly because of its place in the decimal system, is an exception.

The tendency in grouping seems to be to add primarily by tens, then to add by nines which is ten less one. The combinations that form eleven and eight appeared in the adding with nearly the same frequency. The subjects report that eight is often added as ten less two, and that eleven is nearly always added as ten plus one.

The cause of such errors as were made seemed to lie very frequently in the influence of some preceding figure still delaying in the mind. Bergerstein[1] in his studies of Vienna school

[1] *Zeitschrift für Schulgesundheitspflege*, Vol. IV, pp. 556-560.

children, found this to be one of the most common causes of error, both in addition and multiplication. Sometimes the tens were not carried or were carried too far. Excessive combination of the numbers seemed to involve error, in that too much was carried in mind to allow of clear holding of all the elements.

When uncertain, the subjects often tried to verify their results as they went along. The only means of verification used, except that of re-adding, was manipulation of the digital combinations. When there was doubt, for example in combinations such as 37+6, the poor adders would take three from the six and combine it with the 37 to make 40 and then add the three; and in some cases they even counted by units. The more proficient ones would in such a case refer to the digital combination 7 + 6, and getting the result as thirteen, would carry the ten almost mechanically. But no hard and fast line of distinction can be drawn, for at times the best adders would break up the numbers, and the poor ones would refer to the digital combinations, though they could not always recall them.

The unanimous opinion was that it is more difficult to make large than small steps in the adding, *i. e.*, to add the larger digits. Further evidence of this appears in the fact that when the subject was not feeling well, or otherwise was not in the best condition for adding, he made fewer combinations, sought only the easier ones, and if there was any inclination to count, to separate numbers, or to take them out of their natural order; all these devices were resorted to. The reference to the digital combinations was most frequently in visual terms. One subject said that he could "see the work within three or four feet in front of him." The tactual and motor senses as aids in combining have probably a deeper meaning, but are less readily observed in rapid work, except as the hand or pencil is used to keep the place which is perhaps itself an assistance.

Another subject, a high school student, a poor adder, would seek every opportunity to separate the numbers and make easy combinations, and where the numbers were small they were added as units. The gentlemen that made few combinations would most frequently take numbers out of their order to facilitate easy combinations. The seemingly mechanical process of the rapid adders would lead them to forget their exact position with relation to the tens, thus making them liable to err. Excessive grouping of numbers is a loss of time in some cases, as it causes a hesitancy in adding, time being also required to decide what groups are best to make. But on the whole the results would seem to indicate that the subject who can use a few combinations judiciously has the advantage in the process.

From observation in this work it would seem that the trained

apprehension of the sum of two digits is a process not unlike the recognition of the proper pronunciation of a word. When we see a word, we rarely think of the sounds of which it is composed, but grasp it as a whole. To produce a similar degree of proficiency should be the aim of the teacher of elementary arithmetic.

On the Rate of Adding Certain Combinations of Digits. In this section the writer ventures to present the somewhat meagre results of a chronoscopic study of certain special additions. Incidentally determinations were also made of the reading time for the single numbers from 0 to 12. Both series of experiments were made in the usual way with the Hipp chronoscope and mouth key. The numbers (about three-quarters of an inch high) were displayed at the instant of starting the chronoscope by means of a Cattell fall-chronometer used as a falling screen. The uniformity of the chronoscope was tested at the beginning and end of each sitting by means of a large regulating pendulum; and of the subject's reactions only such were preserved in the protocol as at the time satisfied both subject and experimenter as having been made under standard conditions both internal and external.[1]

Two subjects (A and B) participated in the reading experiment and furnished an average of twelve records each for each of the thirteen numbers read. The times for reading these numbers range, for A from 281σ to 380σ, and for B from 255σ to 286σ, and give the following orders of quickness in reading:

For A:	0,	8,	1,	10,	2,	11,	7,	3,	12,	9,	5,	4,	6.
" B:	8,	4,	3,	2,	1,	0,	5,	10,	11,	6,	12,	7,	9.

If we separate each of these series into a quicker and a slower half (above and below the median term—7 for A and 5 for B) the following points of agreement may be noted: First, 0, 1, 2, and 8 for both subjects come before the median (*i. e.*, are quickly read), while 6, 9, and 12 follow the median and are slowly read. The medians themselves (7 for A and 5 for B) belong, each in the case of the other subject, to the slower group and may well be classed with the slower numbers. The remaining numbers 3, 4, 10, and 11, show different results with the two subjects and must be left doubtful. There is of course little probability that these differences are due to differences in the ease of recognition of the individual number symbols or to differences in the facility of the associative processes between the symbol and its name. A much more likely cause is to be found in a varying difficulty of enunciation or in the way in

[1] The chronoscope values have not been reduced to absolute time values as the chief interest in the determinations is relative.

which the mouth-key operated in case of the different movements required in speaking the different names.

The tests on adding followed those on the reading of numbers. Subject A added the combinations found in the tables of 2's, 3's, 7's, and 8's from 0 to 12 complete (2+0, 2+1, 2+2 . . . . 2+12, 3+0, 3+1, 3+2 . . . . 3+12, etc.), presented in irregular order to an average of eleven times each. B added certain selected combinations from the same tables to an average of ten times each. A's times range from 479σ for 2+8 to 655σ for 8+12; B's times from 377σ for 8+10 to 824σ for 8+12; but in A's case nearly four-fifths of the times fall between 479σ and 579σ, and in B's case three-fourths between 419σ and 519σ. The fact that the range of variations in these adding-times are on the whole so little greater than those found for the simple reading of numbers, points out at once the impossibility of making determinations on the relative ease of adding the different combinations of numbers without very much more prolonged and careful experimentation. In fact the introspection of the subjects testified that in many cases the sum was reached by an association which seemed practically as simple as that of reading (*i. e.*, naming the number symbols), and one therefore in which specific differences in time would hardly be expected to appear. It is interesting to notice, however, that the characteristic lengthening of the time for the adding, which was originally the more complex process, still persists, though this again might disappear with sufficiently prolonged practice.

One or two relations are to be found in the full tables with sufficient definiteness to warrant mention. With but a single exception (19 cases in 20) it seems to be a relatively slow process to add to the larger numbers 9, 10, 11 and 12, but how much of this is associational, and how much to be credited to difficult enunciation in the 'teens cannot be said. In seven cases out of eight it seems to be a relatively quick process to perform additions resulting in 10,—but "ten" was again an easy name for one of the subjects to say. The sum of a number added to itself was often more quickly announced than the median combination in the same table—three cases out of four for A, and two out of four for B, with a third standing next the median on the slower side. The instance of failure for A was 3+3 for which he was continually beset to announce the product instead of the sum. The greater ease of such combinations was repeatedly noticed in A's introspections.

In other instances the dicta of introspection were little or not at all supported by the chronoscope. A reported, for example, that even numbers seemed easier to add than odd ones, but the tables show little evidence of it. Both subjects were frequently

conscious of taking the figures in such an order as to bring the larger digit first if there were any considerable difference in size, thus 8 and 3 were added by preference as 8+3 even when presented as 3+8, but the tables again show no lengthening of the time to correspond to such a re-arrangement. In fact one cannot but ask himself whether the conscious re-arrangement may not have followed, rather than preceded, the announcing of the sum—the associative reaction having taken on a characteristically motor form.